The Artin-Rees Lemma + Krull intersection

Let R be a ring and I an ideal. Define the ring

$$R^* := R \oplus I \oplus I^2 \oplus \cdots$$ (Mike will tell us more about this object in his presentation.)

If M is an R-module and $\mathcal{J} : M = M_0 \supset M_1 \supset \cdots$ an I-filtration, then

$$M^* := M_0 \oplus M_1 \oplus \cdots$$

is naturally a graded R^*-module. We will use this module to prove an important fact about ideals. First we need the following:

Lemma: Let each M_i be f.g. Then $\mathcal{J}$ is I-stable $\Leftrightarrow$ M^* is a finitely-generated R^*-module.

Pf: If M^* is f.g., then the generators are contained in the first n terms, some n. Thus, $M_n \oplus M_{n+1} \oplus \cdots$ is generated by M_n, so $M_{n+i} = I^i M_n$ for $i \geq 0$, so $\mathcal{J}$ is stable.

Conversely, assume $\mathcal{J}$ is I-stable. Then $M_{n+i} = I^i M_n$, some n, for all $i \geq 0$. Thus M^* is generated by the union of sets of generators for $M_0, \ldots, M_n$. $\square$

We can now prove the Artin-Rees Lemma:

Artin-Rees Lemma: let R be Noetherian, $I \subseteq R$ an ideal, and $M' \subseteq M$ finitely generated R-modules. If

$$\mathcal{J}: M = M_0 \supseteq M_1 \supseteq \dots$$

is I-stable, then so is

$$\mathcal{J}': M' \supseteq M' \cap M_1 \supseteq \dots$$

Pf: Since I is f.g., R^* is a f.g. R-algebra. Thus, by Hilbert basis, it's Noetherian (since it's a quotient of a poly ring over R).

M^* is a f.g. R^*-module (by the lemma), so M^* is Noetherian. Thus, $M' \oplus M' \cap M_1 \oplus \dots \subseteq M^*$ is f.g., so $\mathcal{J}'$ is I-stable. $\square$

Cor: w/ the same setup as above, set $M_m = I^m M$ $\forall$ m. $\exists$ some n s.t. for all $i \geq n$, $(I^i M) \cap M' = I^{i-n}((I^n M) \cap M')$.

Pf: $(I^i M) \cap M' = M_i \cap M' \underset{\substack{\uparrow \\ \text{stability} \\ \text{of } \mathcal{J}'}}{=} I^{i-n}(M_n \cap M') = I^{i-n}((I^n M) \cap M')$. $\square$

Now we get this important result as an easy corollary:

Krull Intersection Theorem: Let R be Noetherian and $I \subseteq R$ an ideal. If M is a finitely generated R-module, then there is some $r \in I$ s.t.

$$(1-r)\left(\bigcap_{j=1}^{\infty} I^j M\right) = 0.$$

Pf: Let $M' = \bigcap_{j=1}^{\infty} I^j M \subseteq M$. M is f.g. over a Noetherian ring, so it is Noetherian. Thus M' is f.g., so we can apply Artin-Rees:

$M \supseteq IM \supseteq I^2M \supseteq \dots$ is I-stable. Thus,

$M' \supseteq (M' \cap IM) \supseteq \dots$ is I-stable as well.

The corollary says that there's some n s.t.

$$(I^{n+1}M) \cap M' = I\left((I^n M) \cap M'\right).$$

In this case, $M' \subseteq I^{n+1}M \subseteq I^n M$, so this becomes $M' = IM'$. We don't know that I is in the Jacobson radical, so we can't apply Nakayama, but we can apply a corollary of C-H that says $\exists\ r \in I$ s.t. $(1-r)M' = 0$. $\square$

Cor: If R is an integral domain or a local ring and $I \subsetneq R$ a proper ideal, then $\bigcap_{j=1}^{\infty} I^j = 0$.

Pf: Set $M = R$ in the theorem. Then $M' = \bigcap I^j$, and the theorem says $(1-r)(\bigcap I^j) = 0$ for some $r \in I$.

If R is an integral domain, then since $1-r \neq 0$, $\bigcap I^j$ must be 0.

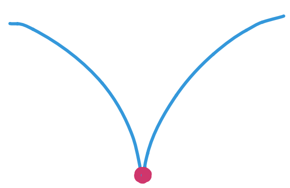
If R is local, then r is in the max'l ideal, so $1-r$ is a unit. Thus $\bigcap I^j = 0$. $\square$

This has a geometric interpretation. Roughly, in the setting of the corollary, if a function vanishes along I to arbitrarily high order, then it's 0.

Ex: $I = (x, y) \subseteq k[x,y]$.

$x \in I$ vanishes on the origin

$x^2 - y^3 \in I^2$ has "second order" vanishing



If $f \in I^n$ for all $n > 0$, then $f = 0$ by KIT.

This can fail for a non integral domain:

Ex: $R = k[x]/(x^2 - x)$

Set $I = (x)$. Then $I^2 = (x^2) = (x)$, so $x \in \bigcap I^j$. i.e. x vanishes to infinite order at the point corr. to (x), but it's not zero, since it's nonzero on the other point.